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EXPONENTIAL GROWTH OF DISTANCE BETWEEN NEARBY RAYS DUE TO MULTIPLE GRAVITATIONAL SCATTERINGS

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ABSTRACT

We give an estimate of the relative error in the angular measurement of observations for high-redshift objects induced by gravitational scatterings (lensing). Gunn concluded in 1967 that the gravitational scatterings by galaxies induce a relative error of a few percent in observations for objects at $z = 1$. This estimate has been considered as a fundamental limitation of accuracy of the angular measurements in observational cosmology. In multiple gravitational scatterings, the bending angle of a single ray grows through the random work process. Gunn assumed that the difference of nearby rays also grows through the random walk process. However, the distance between nearby photons grows exponentially because the two rays suffer coherent scatterings by the same scattering object. This exponential growth continues as long as the scattering is coherent. In the case of scattering by individual galaxies, the exponential growth continues until the angular distance reaches 1′ or so. The relative error of the angular measurements under 1′ due to the exponential growth is $\sim 30\%$ at $z = 1$ and exceeds 100% at $z = 3$, when the density parameter of galaxies is 0.2. The effects of clusters of galaxies or superclusters are more difficult to estimate accurately but might be significant. In the case of superclusters, the angular measurements up to a few degrees could be affected.

Subject headings: cosmology: observations — gravitational lensing

1. INTRODUCTION AND SUMMARY

A photon is gravitationally scattered by galaxies during its travel from its source to us. By this effect of gravitational scattering (lensing), an image of a high-redshift object can be distorted into a double-image and an arc, or can be attenuated (Blandford & Narayan 1992, and references therein). In this *Letter*, we calculate the relative deflection of two photons and investigate change in the observed angle of the high-redshift object induced by the gravitational scatterings. The change in the angle puts a limitation on the angular measurement of the observation of high-redshift objects and gives approximately the change of their size and apparent magnitude. We found that the change due to the gravitational scattering is larger than the estimate of the previous researchers (e.g., Gunn 1967).

Figure 1 shows the basic idea. We observe two photons with the relative angle of θ_o. They are actually originated from points P and Q at $z = z_s$, and the real angular distance between them is θ_s. The trajectories of the photons are deflected by the gravitational scatterings, resulting in $\theta_o \neq \theta_s$. In order to evaluate the difference between θ_o and θ_s, we consider the following situation: The observer "sees" two photons with angle θ_o at present. We calculate the orbit backward in time and determine from where they come. We define a "demagnification" factor, $\alpha(z)$, between two angles as $\alpha(z) \equiv \theta_s(z)/\theta_o$. For two points with angular distance θ_s, the observed angular distance becomes θ_o.

Gunn (1967) calculated analytically the relative deflection between two rays due to the gravitational scattering and concluded that the relative change of observed angle induced by the gravitational scattering is a few percent for the object at redshift $z = 1$. This estimate has been regarded as a fundamental limitation of accuracy of angular measurements in observational cosmology. Kantowski (1969), Dyer & Roeder (1974), and Dyer & Oattes (1988) investigated distortion of distant objects using the Swiss cheese model. Blandford & Jaroszynski (1981) modified the calculation of Gunn (1967) using the newly measured two-point correlation function of galaxies, and Watanabe & Tomita (1991) calculated the relative deflection among rays using the result of the numerical simulation of light propagation by Tomita & Watanabe (1989). They obtained results similar to Gunn's (1967) result.

In the previous theoretical estimates, it was assumed that the two neighboring rays were described by simple behavior of a random walk. However, the effect of multiple scatterings on the neighboring rays cannot be described by a random walk. In fact, the distance between two rays grows exponentially due to multiple gravitational scatterings. In this *Letter* we estimate the change in angle due to the exponential growth.

A basic explanation of the exponential growth of distance is as follows. More detailed discussions have been given by Goodman, Heggie, & Hut (1993). When the distance between two photons is small, the photons are scattered coherently by a single scattering object. In this coherent scattering, the distance increases because the photon closer to the scattering object is deflected with a larger angle. When they are scattered the next time, the distance increases again. The scattering angle θ is inversely proportional to the impact parameter. It is expressed as $\theta \simeq 4GM/(bc)^2$, where G is the gravitational constant, M is the mass of the scattering object, b is the impact parameter, and c is the light velocity. The difference in the scattering angle is given by

$$\Delta\theta = w \frac{d\theta}{db} \simeq \frac{4GM}{b^2c^2} w , \tag{1}$$

where w is the distance between the two photons. Equation (1) implies $\Delta\theta \propto w$. On average, the distance between nearby photons grows exponentially due to the multiple scatterings, as long as the scatterings are coherent. Figure 2 illustrates this

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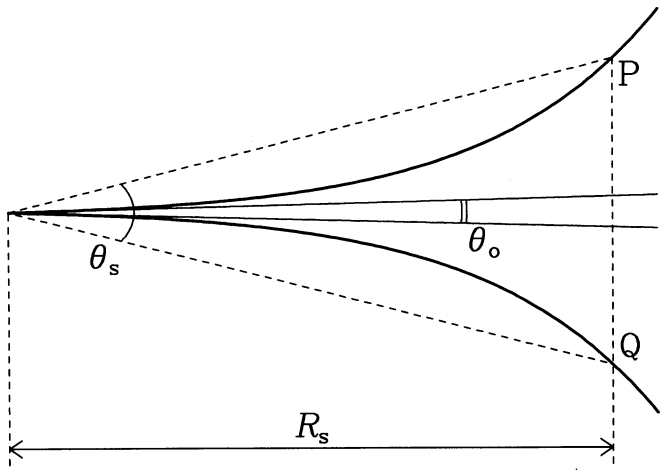

FIG. 1.—Change in angle between two rays emitted from the observer. The angles θ_s and θ_o are the true and observed angular size, respectively.

exponential growth of the distance between two nearby photons.

In the community of N-body simulations, the exponential instability of the initial condition has been well known (Miller 1964; Lecar 1968; Gurzadyan & Savvidy 1986; Sakagami & Gouda 1991; Suto 1990; Kandrup & Smith 1991; Kandrup, Smith, & Willmes 1992; Quinlan & Tremaine 1992; Huang, Dubinski, & Carlberg 1993; Goodman et al. 1993). According to Goodman et al. (1993), the e-folding time of this exponential instability is of the order of 0.1 crossing times, and the exponential growth would saturate when the distance between two orbits becomes $RN^{-1/2}$, where R is the characteristic radius of the system and N is the number of particles in the system. In view of light propagation, the result of Tomita & Watanabe (1989) showed the exponential growth of the initially small angle in the numerical simulation including cosmological expansion. Fukushige, Makino, & Ebisuzaki (1994a) and Fukushige et al. (1994b) discussed the effect of this exponential instability on the observed temperature anisotropy of the cosmic background radiation.

This exponential growth stops when the scatterings become incoherent. The condition for the exponential growth is

$$w < w_{\rm cr} \sim \left(\frac{GM}{c^2 n}\right)^{1/4}, \qquad (2)$$

where $w_{\rm cr}$ is the critical distance that the exponential growth occurs and n is the number density of the scattering objects. If we consider the case of $\Omega_s = 1$, where Ω_s is the density parameter of the scattering objects, the critical distance at $z = 0$ can be written as $w_{\rm cr} \sim R_H N^{-1/2}$, where R_H is the distance to the horizon and N is the number of the scattering objects. After the distance becomes larger than this critical distance, the distance grows diffusively. In this region of the diffusive growth, we should use Gunn's (1967) formula.

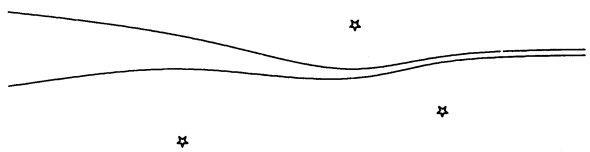

FIG. 2.—Exponential growth of distance between nearby rays due to multiple gravitational scatterings.

In this *Letter* we estimate the change in the angle between two rays due to the gravitational scattering. Our main conclusions are summarized as follows. The demagnification factor defined as the ratio between the true and observed angular size of an object at redshift z is expressed as

$$\alpha(z) \simeq \frac{1}{1 + 1.28\sqrt{\Omega_s}}\left[\frac{(1+z)^{1.92(\Omega_s)^{1/2}} - (1+z)^{-3/2}}{1 - (1+z)^{-3/2}}\right]. \qquad (3)$$

For individual galaxies, $\Omega_s = 0.2$ would be a reasonable estimate. In this case, $\alpha \sim 2$ for redshift $z = 3$, and $\alpha \sim 3$ for $z = 6$. This growth continues until the distance between rays becomes similar to the critical distance $w_{\rm cr}$. For the case of galaxies, $w_{\rm cr} \sim 1'$. This means that the error is as large as 100% in the angular measurement with angular size smaller than 1′, which corresponds to a size of $\sim 0.5\ h^{-1}$ Mpc at $z = 1$, where h^{-1} is the Hubble constant in units of 100 km/(s Mpc).

For this exponential growth to occur, the scattering objects must be more compact than the critical size. Galaxies are compact enough to be effective as the scattering object. The critical size for the scattering objects is the critical distance $w_{\rm cr}$. If the half-mass radius of the scattering object, r_h, is smaller than the critical distance $w_{\rm cr}$, the exponential growth occurs. This is because the main contribution of the exponential growth is due to the scatterings having impact parameters of the order of $w_{\rm cr}$ (Goodman et al. 1993). Numerical calculations confirmed the saturation of the growth factor at larger r_h than the critical distance of $w_{\rm cr}$ (Fukushige et al. 1994a, b). The critical distance $w_{\rm cr}$ of galaxies is estimated as ~ 0.5 Mpc in the case of mean separation of galaxies: $d_{\rm sep} = 3$ Mpc and $\Omega_s = 0.2$. Therefore, the exponential growth due to the gravitational scattering by galaxies occurs, unless a dark halo of galaxies spreads up to a size of ~ 0.5 Mpc.

None of the previous studies took into account this exponential growth of the distance between rays. They estimated the effect of the multiple scatterings by a superposition of the single scatterings. For example, Gunn (1967) calculated the observed relative deflection, $\boldsymbol{n}$, between two rays, using the expression

$$\boldsymbol{n} = 4GM \int \left[\frac{\boldsymbol{D} + \boldsymbol{d}}{(D+d)^2 + \varepsilon^2} - \frac{\boldsymbol{D}}{D^2 + \varepsilon^2}\right]\frac{l_{\rm ex}}{l_{e0}}\, dN(x), \qquad (4)$$

in equation (12) of his paper, where D is the impact parameter, d is the difference of the impact parameter, G is the gravitational constant, ε is the size of the scattering objects, $dN(x)$ is the number of galaxies at x, and $l_{\rm ex}$ is the distance from the emitter to x. In equation (4) the total effect of the multiple scatterings is calculated as simple summation of many scatterings with the distance between two rays, d, unchanged. Therefore, the possibility of the exponential growth is precluded in equation (4).

2. CHANGE OF ANGLE DUE TO THE GRAVITATIONAL SCATTERING

In this section, we estimate the demagnification factor $\alpha(z)$. Here, we assume that the universe is flat ($\Omega = 1$) for simplicity. The angle $\theta_s(z)$ is given by $\theta_s(z) = w(z)/R_s$, where $w(z)$ is the distance between photons at redshift z, and R_s is the distance from $z = 0$ to redshift z. The evolution of the distance w is approximately described by the equation:

$$\frac{dw(z)}{dt} \simeq -c\theta_o - \frac{w(z)}{t_e(z)}, \quad w(z=0) = 0, \qquad (5)$$

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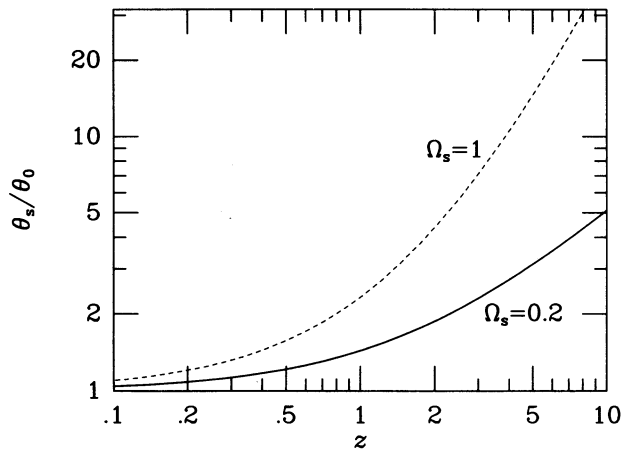

FIG. 3.—Demagnification factor $\alpha(z)$ as a function of redshift z. The solid curve is for the case of $\Omega_s = 0.2$, where Ω_s is the density parameter of galaxies, and the dotted curve is for $\Omega_s = 1$.

where t_e is the e-folding time of the exponential growth of the distance between two rays. The e-folding time t_e is given by $t_e = \tau/(G\rho)^{1/2}$, where ρ is the mass density of the scattering objects and $\tau = 0.18$, according to the numerical simulations of Fukushige et al. (1994a). In an expanding universe, the e-folding time is expressed as

$$t_e(z) = \tau\left(\frac{8\pi}{3H_0^2\Omega_s}\right)^{1/2}(1+z)^{-3/2}, \tag{6}$$

where H_0 is the Hubble constant. By solving equation (5) with equation (6), we obtain $\theta_s(z)$ as

$$\theta_s(z) \simeq \frac{\theta_o}{1+\eta}\,\frac{(1+z)^{3\eta/2} - (1+z)^{-3/2}}{1-(1+z)^{-3/2}}, \tag{7}$$

where $\eta = (\Omega_s)^{1/2}(6\pi)^{-1/2}\tau^{-1}$. Figure 3 shows the demagnification factor $\alpha(z)$ as a function of redshift z for $\Omega_s = 0.2$ (the solid curve) and $\Omega_s = 1$ (the dashed curve). In the case of $\Omega_s = 0.2$, the angle at $z = 0$ increases by a factor 2 at $z = 2$ and by a factor 3 at $z = 5$. If total mass of the universe is in galaxies, that is, $\Omega_s = 1$, $\alpha \simeq 5$ at $z = 2$ and $\alpha \simeq 15$ at $z = 5$.

Figure 4 shows $\theta_s(z)$ as a function of z for different values of

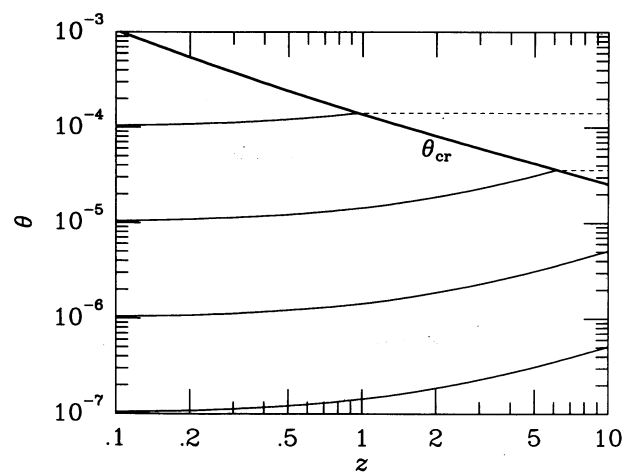

FIG. 4.—True angular size $\theta_s(z)$ for different observed angles at $z = 0$, plotted as a function of z. The density parameter is $\Omega_s = 0.2$. The dashed curve indicates the true angular size, $\theta_{cr}(z)$, of the object with the size w_{cr} at redshift z. If $\theta < \theta_{cr}$, the change of θ is described by the exponential growth.

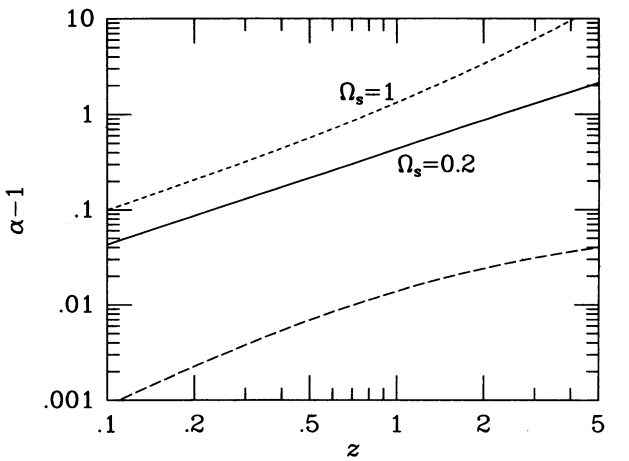

FIG. 5.—Error $[\{[\theta_s(z) - \theta_o]^2/\theta_o^2\}^{1/2}]$ in angular measurement on observation for high-redshift objects. The solid and dotted curves show our estimate for the case of $\Omega_s = 0.2$ and $\Omega_s = 1$, respectively. The dashed curve shows the error estimated by Gunn (1967) for $\Omega_s = 1$.

θ_o for the case of $\Omega_s = 0.2$ and $d_{sep} = 3\ h^{-1}$ Mpc. The thick curve shows the angular size, $\theta_{cr}(z)$, of the object with the size equal to the critical distance w_{cr}. The angle $\theta_{cr}(z)$ is given by

$$\theta_{cr}(z) \simeq \frac{w_{cr}}{R_s(z)} \simeq 20''\Omega_s^{-1/4}\,\frac{(1+z)^{-3/4}}{1-(1+z)^{-3/2}}. \tag{8}$$

If $\theta < \theta_{cr}$, the change of θ is given by equation (7); if $\theta > \theta_{cr}$, the angle θ changes by the random walk.

Gunn's (1967) estimate of the error in the angular measurements, given in his equation (41a), is

$$\left[\frac{(\theta_s-\theta_o)^2}{\theta_o^2}\right]^{1/2} \simeq \left(\frac{8C}{30}\right)^{1/2}[1-(1+z)^{-1/2}]^{3/2}, \tag{9}$$

where $C = 2.9 \times 10^{-2}$. Figure 5 shows the errors in the angular measurement as a function of redshift z. The dashed curve is the error estimated by Gunn (1967) for $\Omega_s = 1$. Our estimate is larger than that of Gunn (1967) by more than two orders of magnitude.

3. DISCUSSION

We estimated the effect of the gravitational scattering using the factor α between the true and observed angular size. Using this factor α, we can approximately estimate the increase of the apparent magnitude of the high-redshift sources due to gravitational scattering. Since the surface brightness of the source does not change with gravitational scattering, the apparent magnitude is determined by the angular size of the source. If we assume that the size of a typical object shrinks by a factor $1/\alpha(z)^2$, the change Δm in the apparent magnitude m is

$$\Delta m = m_0 - m = \frac{5}{2}\log_{10}\alpha(z)^2$$

$$= \begin{cases} 5\log_{10}\left[\dfrac{(1+z)^{0.86} - (1+z)^{-3/2}}{1-(1+z)^{-3/2}}\right] - 0.98, & (\Omega_s = 0.2) \\ 5\log_{10}\left[\dfrac{(1+z)^{1.92} - (1+z)^{-3/2}}{1-(1+z)^{-3/2}}\right] - 1.79, & (\Omega_s = 1), \end{cases} \tag{10}$$

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where m_0 is the apparent magnitude for the case without the gravitational scattering. The apparent magnitude of a source at $z = 5$ increases by 2.5 for $\Omega_s = 0.2$ and by 5.8 for $\Omega_s = 1$. This effect certainly affects the magnitude-redshift relation. Also, this effect may be related to cutoff in the distribution of quasars at high z.

Finally, we estimate the effect of clusters of galaxies and superclusters. Both clusters of galaxies and superclusters are compact enough to work effectively as the scattering object of the exponential growth. The APM survey for clusters of galaxies (Dalton et al. 1992) showed that the mean separation of clusters is 45 h^{-1} Mpc. The thickness of the Great Wall in the CfA survey (Gellar & Huchra 1989) is 5 h^{-1} Mpc, while the size of a void is ~ 100 h^{-1} Mpc. However, the density parameter of a cluster of galaxies and superclusters has a large uncertainty. Also, the formation time of these objects, or their very existence, is still controversial. Because of the larger average separation, the critical angle θ_{cr} is larger for clusters of galaxies or superclusters than for galaxies. For superclusters, θ_{cr} is as large as a few degrees.

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